REFLECTION AND REFRACTION OF A PLANE PLASTIC WAVE IN THE PRESENCE OF A BOUNDARY PLANE

PMM Vol. 31, No. 5, 1967, pp. 848-860
N. V. ZVOLINSKII
(Moscow)
(Received April 25, 1967)

One-dimensional problems of plane plastic wave reflection were examined in a number of papers [1 to 12]. Nevertheless, the problem cannot be considered exhausted. Even in the comparatively simple case of explosive loading and propagation in the shock wave regime there are unsolved problems. Published papers either contain solutions based on the simplest approximation of the compression relationship, or lead to complex analytical descriptions from which sometimes no conclusions are drawn. Numerical methods are also little developed in this area.

Some general properties of the reflection problem are studied below under the assumption of rigid unloading. The nature of phenomena is also investigated on the basis of numerical solutions.

In connection with this it turned out that the a priori assumption, made by many authors including papers [1 to 3] about unloading taking place in the reflected wave region, is in general mistaken. It is true that errors arising from this mistake are usually small. In particular, if the compression relationship is linear, the hypothesis of unloading is justified. An influence of the nature of unloading on this effect undoubtedly exists. This influence is not examined in this paper.

The influence on propagation of a reflected wave due to a boundary layer with a given stress is investigated. The reflected wave begins to "feel" the external loading immediately after the start of reflection. As is brought out in this paper, the influence is small at first, but it increases gradually, reaches a determining value and finally leads to the annihilation of the shock wave which can never reach the boundary plane with the exception of the case of a stationary wave. It is proved here that this fact, noted for particular cases in [6, 7 and 11], is general in nature. In papers [1 to 3] the problem of reflection was solved without taking the boundary plane into consideration. It is shown here that such a solution has a limited character and with accuracy to small terms of second order describes the asymptotic behavior of the phenomenon at instants near the beginning of reflection. Reflection was investigated earlier in papers [5 to 7 and 12] taking into account the boundary plane for simple compression relationships.

1. Two media, the first of which fills a plane-parallel layer and second a half-space, are in contact along the plane. We shall examine one-dimensional plane motions polarized perpendicular to this plane. The uniaxial compression diagram of the first medium is assumed to have the form presented in Fig. 1. The regime of shock waves is investigated below. A working region, which is assumed to be concave upward, above point A corresponds to this regime. Experimental facts show compressibility of solid media (soils, metals and others) at quite high pressures. Therefore the presence of a vertical asymptote

is not assumed in the compression diagram. Unloading and repeated loading occur with conservation of particle density. The second medium is either plastic and is then described by an analogous compression diagram, or linearly elastic. The equations of the working part of these diagrams will be written in the form

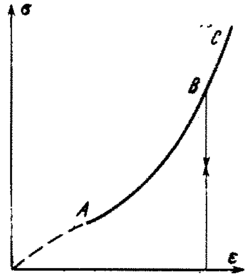

Fig. 1

$$\sigma = \sigma_1^\circ f_1(\varepsilon), \qquad \sigma_1^\circ > 0 \quad \text{(for the first medium)}$$

$$\sigma = \sigma_2^\circ f_2(\varepsilon), \qquad \sigma_2^\circ > 0 \quad \text{(for the second medium)}$$

Here σ_1°, and σ_2° are constant coefficients having the dimensions of stress.

We shall designate $a_1 = \sigma_1^\circ / \rho_1$ and $a_2 = \sigma_2^\circ / \rho_2$, where ρ_1 and ρ_2 are the densities of media. The stress of compression and deformation of compression are assumed to be positive.

The motion will be described in Lagrangian coordinates h and t. In this case the axis h is oriented perpendicularly to the interface of media ($v(h, t)$ is the particle velocity). The motion is described by Eqs.

$$\frac{\partial \sigma}{\partial h} + \rho \frac{\partial v}{\partial t} = 0, \qquad \frac{\partial v}{\partial h} + \varepsilon'(\sigma) \frac{\partial \sigma}{\partial t} = 0 \tag{1.1}$$

Here $\varepsilon = \varepsilon(\sigma)$ is the inversion of the relationship $\sigma = \sigma(\varepsilon)$, in particular $\varepsilon'(\sigma) = 0$ in the regime of unloading. In the region in which unloading takes place $\varepsilon = \text{const}$, therefore we have there

$$v = v(t), \qquad \sigma(h, t) = -\rho v'(t) h + C \tag{1.2}$$

On the shock wave two mechanical conservation conditions are satisfied

$$v_- - v_+ = (\varepsilon_- - \varepsilon_+) h'(t), \qquad \sigma_- - \sigma_+ = \rho(v_- - v_+) h'(t) \tag{1.3}$$

quantities designated by the subscript plus refer to condition ahead of the front, quantities designated by the subscript minus to conditions behind the front: $h(t)$ denotes the coordinate of the front of the shock wave.

Let the interface of media be the plane $h = h_0$. Assuming complete contact, we require continuity of stresses and displacements on this plane

$$\sigma_1(h_0, t) = \sigma_2(h_0, t), \qquad v_1(h_0, t) = v_2(h_0, t) \tag{1.4}$$

subscripts 1 and 2 designate the sides of the interface.

2. Let the first medium fill the layer $0 \le h \le h_0$, while an external stress $\sigma_0(t)$ is applied on the plane $h = 0$. In this case

$$\sigma_0(0) \neq 0, \ \sigma_0(t) > 0, \ \sigma_0'(t) < 0 \qquad \text{for} \qquad t \geqslant 0$$

If $\sigma_0(0)$ is sufficiently large, then from the boundary plane a shock front will start to propagate behind which the particles will be in the condition of unloading. Designating the coordinate of the front by $h_*(t)$ we obtain the following equations [2] for the description of the incident wave

$$v = \varepsilon_* h_*', \quad a_1^2 f_1[\varepsilon(h_*)] = \varepsilon_* h_*'^2, \quad h_* h_*' \varepsilon_* = \frac{1}{\rho_1} \int_0^t \sigma_0(\tau) d\tau, \quad h_*(0) = 0 \tag{2.1}$$

If the second medium is plastic, then under the condition that the wave incident on the interface is sufficiently intense a shock wave will propagate also in the second medium.

If the second medium is ideally elastic, we obtain for the particle velocity and stress

$$v_2(h, t) = V\left(t - \frac{h - h_0}{a_2}\right), \qquad \sigma_2(h, t) = \rho_2 a_2 v_2(h, t) \tag{2.2}$$

Here $V(t)$ denotes $V(t) = v_2(h_0, t)$, while a_2 is the velocity of the elastic wave in the second medium.

3. Let us assume that the second medium in some respect is more rigid than the first (see Section 5 below). In such a case a reflected shock wave will start to propagate from the interface in the opposite direction. The layer $0 \leq h \leq h_0$ is divided by this wave into two parts

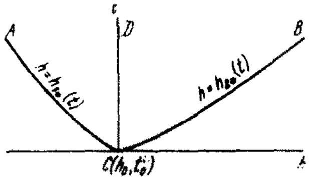

Fig. 2

$$h_{1*}(t) \leqslant h \leqslant h_0 \quad \text{(region 1)}$$

$$0 \leqslant h \leqslant h_{1*}(t) \quad \text{(region 3)}$$

No a priori assumptions whatsoever are made with respect to the regime in region 1. In region 3 the loading and unloading of particles takes place with conservation of density: ahead of the reflected wave front that stress $\sigma_b(h)$ is attained which was the greatest for the corresponding particle in the incident wave. This independent hypothesis is connected with the requirement of stability of the shock wave (see e.g. [1]). Therefore we have in region 3

$$v_3(h, t) = v_3(t), \qquad \sigma_3(h, t) = -\rho_1 v_3'(t)\, h + \sigma_0(t) \tag{3.1}$$

Here the boundary condition on the plane $h = 0$ is already taken into account. Since $\sigma_3(h_{1*}, t) = \sigma_b, (h_{1*})$, then

$$v_3'(t) = \frac{\sigma_0(t) - \sigma_b(h_{1*})}{\sigma h_{1*}(t)} \quad \text{or} \quad v_3(t) = v(t_0) + \frac{1}{\rho}\int_{t_0}^{t} \frac{\sigma_0(\tau) - \sigma_b(h_{1*})}{h_{1*}(\tau)}\, d\tau \tag{3.2}$$

The initial condition $v_3(t_0) = v(t_0)$ follows from the theorem on the amount of motion for a mass belonging to region 3 at the instant $t_0 + 0$. Conditions (1.3) applied to the front of the reflected wave generate the following Eqs.:

$$f_1[\varepsilon_1(h_{1*})] - f_1[\varepsilon(h_{1*})] = \frac{(v_1 - v_3)\, h_{1*}'}{a_1^2}, \qquad \varepsilon_1(h_{1*}) - \varepsilon(h_{1*}) = \frac{v_1 - v_3}{h_{1*}'} \tag{3.3}$$

The same conditions for the transmitted wave give

$$f_2[\varepsilon_2(h_{2*})] = v_2 h_{2*}' / a_2^2, \quad \varepsilon_2(h_{2*}) = v_2 / h_{2*}' \tag{3.4}$$

The following boundary value problem arises for quasi-linear systems of first order equations of hyperbolic type. Two regions in the plane h, t (Fig. 2) are examined.

ACD bounded by the straight line $h = h_0$ and line $h = h_{1*}(t)$

BCD bounded by the straight line $h = h_0$ and line $h = h_{2*}(t)$

It is required to find functions

$\sigma_1(h, t)$ and $v_1(h, t)$ in region ACD, $\quad \sigma_2(h, t)$ and $v_2(h, t)$ in region BCD

and also functions $h_{1*}(t)$ and $h_{2*}(t)$ for $t > t_0$ under the following conditions.

Functions $\sigma_1(h, t)$ and $v_1(h, t)$ satisfy Eqs.

$$\frac{\partial \sigma_1}{\partial h} + \rho_1 \frac{\partial v_1}{\partial t} = 0, \qquad \frac{\partial v_1}{\partial h} + \varepsilon_1'(\sigma_1) \frac{\partial \sigma_1}{\partial t} = 0 \qquad (h, t) \in ACD$$

and boundary conditions (3.3) on the unknown line AC.

Functions $\sigma_2(h, t)$ and $v_2(h, t)$ satisfy Eqs.

$$\frac{\partial \sigma_2}{\partial h} + \rho_2 \frac{\partial v_2}{\partial t} = 0, \qquad \frac{\partial v_2}{\partial h} + \varepsilon_2'(\sigma_2)\frac{\partial \sigma_2}{\partial t} = 0 \qquad (h, t) \in BCD$$

and boundary conditions (3.4) on the unknown line BC.

For $h = h_0$ the desired functions are connected through boundary conditions (1.4): function $v_3(t)$ is given by Eq. (3.2). Functions $\varepsilon(h)$ and $\sigma_b(h)$ are assumed to be known from the previously obtained solution of the problem on the incident wave. It is natural that the solution of a similar problem in any kind of a simple form is possible only in quite special cases.

However, without solving the problem completely we can draw some qualitative conclusions about the properties of its solution.

a) From conditions (3.3) we obtain

$$\left(\frac{h_{1*}}{a_1}\right)^2 = \frac{f_1[\varepsilon_1(h_{1*})] - f[\varepsilon(h_{1*})]}{\varepsilon_1(h_{1*}) - \varepsilon(h_{1*})} \tag{3.5}$$

The right-hand side is a monotonously increasing function of ε_1, for a fixed value of ε. This follows from the fact that $f_1''(\varepsilon) > 0$ in the investigated region of the diagram. The difference $\varepsilon_1(h) - \varepsilon(h)$ is a monotonously increasing function of h. This follows from the presence of irreversible losses in the shock wave. Therefore

$$\varepsilon_1(h) - \varepsilon(h) < \varepsilon_1(h_0) - \varepsilon(h_0) = \Delta_0$$

As a result we have estimates for the velocity of the front of the reflected wave

$$f_1'[\varepsilon(h_{1*})] \leqslant \left(\frac{h_{1*}'}{a_1}\right) \leqslant \frac{f_1[\varepsilon(0) + \Delta_0] - f_1[\varepsilon(0)]}{\Delta_0} = C \tag{3.6}$$

We can assume in an approximation as it was done in [2] that

$$\frac{h_{1*}'}{a_1} = -\sqrt{f'[\varepsilon(h_{1*})]}$$

For the time of wave propagation from the interface to the particle with a coordinate h_{1*} we obtain

$$t = t_0 + \frac{1}{a_1}\int_{h_{1*}}^{h_0} \frac{dh}{\sqrt{f'[\varepsilon(h)]}} \qquad (h_{1*} < h_0)$$

If the incident wave continued to propagate at $t > t_0$, without encountering the interface, we would have for it

$$t = t_0 + \frac{1}{a_1}\int_{h_0}^{h_*} \left(\frac{\varepsilon(h)}{f[\varepsilon(h)]}\right)^{1/2} dh \qquad (h_* > h_0)$$

Eliminating t we obtain directly an approximate relationship between h_{1*} and h_* (see Eq. (2.5) in [2]).

b) If the incident wave is nonstationary, the reflected shock wave cannot reach the interface ; it is exhausted before it reaches this boundary. We note that $v_1 \geq 0$, for $h_{1*} \leq h \leq h_0$.

In fact

$$\frac{\partial v_1}{\partial h} = -\varepsilon_1'(\sigma_1)\frac{\partial \sigma_1}{\partial t} \leqslant 0, \qquad \frac{\partial v_2}{\partial h} = -\varepsilon_2'(\sigma_2)\frac{\partial \sigma_2}{\partial t} \leqslant 0, \qquad v_{2*} = \varepsilon_{2*} h_{2*}' > 0$$

Since $v_1 = v_2$ for $h = h_0$ the inequality $v_1 > 0$ follows from here. If the reflection takes place from a rigid wall, then $v_1 \equiv 0$. In Eq. (3.2) we assume h_{1*} as the variable of integration. Then

$$v_3(t) = v(t_0) - \frac{1}{\rho_1}\int_{h_{1*}}^{h_0} \frac{\sigma_b(h) - \sigma_0[t(h)]}{h\,|h_{1*}'|}\, dh$$

Since

$$|h'_{1*}| \leqslant aC, \qquad \sigma_b(h) - \sigma_0[t(h)] > \sigma_b(h_0) - \sigma_0(t_0)$$

then

$$v_3(t) \leqslant v(t_0) - \frac{\sigma_b(h_0) - \sigma_0(t_0)}{\rho_1 a_1 C} \int_{h_{1*}}^{h_0} \frac{dh}{h}$$

Taken over the interval 0 to h_0, the integral in this equation is divergent; it follows from this that the difference $v_1 - v_3$ is initially negative, this follows from (3.3), and becomes zero before the reflected wave reaches the boundary plane $h = 0$. The shock wave ceases to exist at the same instant. The layer $0 \leq h \leq h_0$ "hardens" and its stressed condition subsequently follows the pattern of change of external stress.

c) For the incident wave the following relationship is applicable

$$\frac{dv}{dt} = -\frac{\sigma(h_*, t) - \sigma_0(t)}{\rho h_*}$$

For the "precursor" of the reflected wave we have (3.2). From this it follows that

$$\left(\frac{dv}{dt}\right)_{t=t_0} = \left(\frac{dv_3}{dt}\right)_{t=t_0}$$

However, in addition to this

$$(v)_{t=t_0} = (v_3)_{t=t_0}$$

or

$$v_3(t) = v(t_0) + v'(t_0)(t - t_0) + O[(t - t_0)^2] \tag{3.7}$$

In other words, in the beginning of the reflection the field of velocities of the "precursor" coincides with the velocity field of the incident wave with accuracy to small terms of second order (with respect to time). In this connection, speaking about the incident wave, we examine its motion which would take place in the absence of the interface.

4. The incorrectness of the hypothesis that in the reflected and refracted waves unloading begins immediately after the start of reflection can be shown under the following condition

$$\frac{f_1(\varepsilon_0)}{\varepsilon_0} < \frac{f_1(\varepsilon) - f_1(\varepsilon_0)}{\varepsilon - \varepsilon_0} \qquad (\varepsilon_0 < \varepsilon) \tag{4.1}$$

which is characteristic for the regime of shock waves and is applicable in the upward concave region of the compression diagram.

The proof will be made starting with the opposite assumption that the unloading hypothesis applies.

If the second medium is also plastic, then

$$v_1(h, t) = v_1(h_0, t) = V(t), \quad v_2(h, t) = v_2(h_0, t) = V(t) \tag{4.2}$$

It remains to satisfy four conditions (3.3) and (3.4) on the shock waves and the first condition (1.4) on the plane of contact. For the determination of five functions $\varepsilon_1(h)$, $h_{1*}(t)$, $V(t)$, $\varepsilon_2(h)$ and $h_{2*}(t)$ we obtain five Eqs.

$$f_1(\varepsilon_1(h_{1*})) - f(\varepsilon(h_{1*})) = \frac{(V - v_3)h'_{1*}}{a_1^2}, \qquad \varepsilon_1(h_{1*}) - \varepsilon(h_{1*}) = \frac{V - v_3}{h'_{1*}}$$

$$f_1(\varepsilon_1(h_{1*})) + \frac{V'(h_{1*} - h_{2*})}{a_1^2} = \frac{\rho_2 a_2}{\rho_1 a_1} \frac{Vh'_{2*}}{a_2^2} \tag{4.3}$$

$$Vh'_{2*} = a_2^2 f_2(\varepsilon_2(h_{2*})), \qquad V = \varepsilon_2(h_{2*})\, h'_{2*}$$

If the second medium is elastic, the last two Eqs. of the system (4.3) disappear and the third takes the form

$$f_1(\varepsilon_1(h_{1*})) + \frac{V'(h_{1*} - h_0)}{a_1^2} = \frac{\rho_2 a_2}{\rho_1 a_1} \frac{V}{a_1}$$

It will be shown that the system of equations (4.3) is contradictory (*) as a result of the unloading hypothesis. For this purpose let us examine the area near the beginning of reflection. Let H_0 be the distance from the reflecting plane to that plane on which the stress at $t = t_0$, being linearly extended, becomes zero

$$H_0 = \frac{\sigma_b(h_0)}{\sigma_b(h_0) - \sigma_b(t_0)} h_0$$

We introduce the nondimensional parameter $\tau = (t - t_0)a_1 / H_0$. The desired functions are assumed to be in the form of asymptotic expansions:

for the incident wave

$$\frac{h_* - h_0}{H_0} = h'\tau + h''\tau^2 + \cdots, \quad \varepsilon(h) = \varepsilon_0 + \varepsilon' \frac{h - h_0}{H_0} + \cdots$$

$$\frac{v(t)}{a_1} = v^\circ + v'\tau + \cdots, \quad \frac{\sigma_0(t)}{\rho a_1^2} = \sigma^\circ + \sigma'\tau \tag{4.4}$$

for reflected wave

$$\frac{h_{1*} - h_0}{H_0} = h_1'\tau + h_1''\tau^2 + \cdots, \quad \varepsilon_1(h_{1*}) = \varepsilon_{10} + \varepsilon_1'\tau + \cdots$$

$$\frac{V(t)}{a_1} = V^\circ + V'\tau + \cdots \tag{4.5}$$

for transmitted wave

$$\frac{h_{2*} - h_0}{H_0} = h_2'\tau + h_2''\tau^2 + \cdots, \qquad \varepsilon_2(h_{2*}) = \varepsilon_{20} + \varepsilon_2'\tau + \cdots \tag{4.6}$$

Terms which were not written out have the order $O(\tau)$ or $O(\tau^2)$, respectively.

From Eqs. (2.1) we find the coefficients of expansions (4.4)

$$v^\circ = \sqrt{\varepsilon_0 f_1(\varepsilon_0)}, \qquad h' = \sqrt{\varepsilon_0^{-1} f_1(\varepsilon_0)}, \qquad v' = \frac{H_0}{h_0}(\sigma^\circ - f_1(\varepsilon_0))$$

$$\varepsilon' = \frac{2\varepsilon_0 v'}{\varepsilon_0 f_1'(\varepsilon_0) + f_1(\varepsilon_0)}, \qquad h'' = \frac{v'}{2\varepsilon_0} \frac{\varepsilon_0 f'(\varepsilon_0) - f(\varepsilon_0)}{\varepsilon_0 f_1'(\varepsilon_0) + f_1(\varepsilon_0)} \tag{4.7}$$

From here it is evident, in particular, that $v' \leqslant 0$, $\varepsilon' \leqslant 0$, $h'' \leqslant 0$, i.e. the velocity of particles, the velocity of the front and deformation of the incident wave decrease (and not increase) with the extent of propagation.

From (4.5), (4.6) and the system (4.3) we obtain Eqs. for the first approximation

$$f_1(\varepsilon_{10}) - f_1(\varepsilon_0) = (V^\circ - v^\circ) h_1', \qquad (\varepsilon_{10} - \varepsilon_0) h_1' = V^\circ - v^\circ$$

$$f_1(\varepsilon_{10}) = (\rho_2 / \rho_1) V^\circ h_2', \quad f_2(\varepsilon_{20}) = (a_1 / a_2)^2 V^\circ h_2', \quad \varepsilon_{20} h_2' = V^\circ \tag{4.8}$$

and the Eqs. of the second approximation

$$\varepsilon_1' f_1'(\varepsilon_{10}) - \varepsilon' h_1' f_1'(\varepsilon_0) = (V^\circ - v^\circ) 2h_1'' + (V' - v') h_1'$$

$$(\varepsilon_{10} - \varepsilon_0) 2h_1'' + (\varepsilon_1' - \varepsilon' h_1') h_1' = V' - v'$$

$$\varepsilon_1' f_1'(\varepsilon_{10}) + V' h_1' = (\rho_2 / \rho_1) V h_2' + V^\circ 2h_2'' + V' h_2' \tag{4.9}$$

$$\varepsilon_2' f_2'(\varepsilon_{20}) = (a_1 / a_2)^2 (V' h_2' + V^\circ 2h_2''), \qquad \varepsilon_2' h_2' + \varepsilon_{20} 2h_2'' = V'$$

If the compression relationship is linear in the second medium, particularly in the case of loading and if the second medium is ideally elastic, the third Eqs. in systems (4.8) and

*) A system of equations in general form analogous to (4.3) was first presented in [8].

(4.9) are replaced correspondingly by

$$f_1(\varepsilon_{10}) = \frac{\rho_2 a_2}{\rho_1 a_1} V^\circ, \qquad \varepsilon_1' f_1'(\varepsilon_{10}) + V' h_1' = \frac{\rho_2 a_2}{\rho_1 a_1} V'$$

In connection with this the last two Eqs. disappear in (4.8) and also in (4.9). Let us turn to system (4.9). From the last two Eqs. we can find the quantity h_2'':

$$h_2'' = k_0 V', \qquad k_0 = \frac{\varepsilon_{20} f_2'(\varepsilon_{20}) - f_2(\varepsilon_{20})}{2\varepsilon_{20}[\varepsilon_{20} f_2'(\varepsilon_{20}) + f_2(\varepsilon_{20})]}, \qquad |2k_0\varepsilon_{20}| \leqslant 1$$

and as a result the third Eq. in system (4.9) takes the form

$$\varepsilon_1' f_1'(\varepsilon_{10}) + V' h_1' = cV' \qquad (c > 0) \tag{4.10}$$

Here

$$c = \begin{cases} [(\rho_2/\rho_1)(2k_0\varepsilon_{20} + 1) + 1] h_2', & \text{if the second medium is plastic} \\ \rho_2 a_2 / \rho_1 a_1 & \text{if the second medium is elastic} \end{cases}$$

For the determination of unknown coefficients ε_1', h_1'' and V' in both cases a system of equations consisting of the first two Eqs. (4.9) and Eq. (4.10) can be used. The determinant Δ of this system is

$$\Delta = -2(\varepsilon_{10} - \varepsilon_0)\{(c - h_1')[h_1'^2 + f_1'(\varepsilon_{10})] - 2h_1' f_1'(\varepsilon_{10})\} < 0$$

Consequently, the equations can be solved and we obtain for V'

$$V' = -\frac{4 f_1'(\varepsilon_{10})\varepsilon_0}{\Delta} \frac{[f_1(\varepsilon_{10}) - f_1(\varepsilon_0)] - f(\varepsilon_0)(\varepsilon_{10} - \varepsilon_0)}{\varepsilon_0 f_1'(\varepsilon_0) + f(\varepsilon_0)}$$

From this it is evident that $V' > 0$; from (4.10) it follows that also $\varepsilon_1' > 0$. For the stress on the reflecting surface we have

$$\sigma_1(h_0, t) = \sigma_1(h_0, t_0) + \rho_1 a_1^2 c V' \tau + \ldots \tag{4.11}$$

Eq. (4.11) shows that in spite of the assumption about unloading the stress on the contact boundary increases. The stress on the shock wave and consequently in the entire region 1 also increases. This analysis leaves the question open as to the character of the phenomenon in the case when the compression relationship in the first medium is linear under loading. If the second medium is elastic, the hypothesis of unloading is justified. Indeed, the incident wave is described in closed form

$$f_1(\varepsilon) = \varepsilon, \quad h_*' = a_1, \quad \varepsilon(t) = \frac{I(t)}{\rho_1 a_1^2 t} \qquad \left(I(t) = \int_0^t \sigma_0(\tau)\, d\tau\right) \tag{4.12}$$

For the reflected wave we obtain $h_{1*} = -a_1$ and the third Eq. of system (4.3) takes the form

$$(t - t_0) V'(t) + \varkappa_1 V(t) = F(t) \qquad (t_0 \leqslant t) \tag{4.13}$$

$$F(t) = a_1 \varepsilon(2t_0 - t) + v_3(t), \qquad \varkappa_1 = \rho_2 a_2 / \rho_1 a_1 + 1$$

Since $\varkappa_1 > 0$, it is necessary to integrate with the initial condition $|V(t_0)| < +\infty$. We obtain

$$V(t) = (t - t_0)^{-\varkappa_1} \int_{t_0}^t F(t)(t - t_0)^{\varkappa_1 - 1}\, d\tau \tag{4.14}$$

Let function $F(\tau)$ be monotonously decreasing. In fact

$$a_1\rho_1 F(t) = \frac{I(2t_0 - t)}{2t_0 - t} + \rho_1 a_1 v(t_0) - \int_{t_0}^t \frac{[I(2t_0 - \tau)/(2t_0 - \tau)] - \sigma_0(\tau)}{2t_0 - \tau}\, d\tau$$

$$a_1\rho_1 F'(t) = \frac{\sigma_0(t) - \sigma_0(2t_0 - t)}{2t_0 - t}$$

Since $\sigma_0(t)$ decreases monotonously, $F(t)$ also decreases monotonously. If (4.14) is transformed into the form

$$V(t) = \int_0^1 F(t_0 + s(t - t_0))\, s^{\varkappa_1 - 1} ds$$

it becomes apparent that $V(t)$ decreases together with $\sigma_1(h_0, t)$. An analogous proof can be carried out in case when the second medium is plastic but linear under loading.

5. Let us examine the particular case when the incident wave is stationary and has the shape of a step. Expansions (4.4) to (4.6) are reduced to their first terms and give an exact solution of the problem. In this case it turns out that the reflected and transmitted waves are also stationary. The problem is reduced to solution of system (4.8). The possibility of unique solution of this system is also necessary for the justification of arguments in Section 4, since the representation of desired functions in the form of Expressions (4.5) and (4.6) is based on this solution. First let us dwell on the case where the second medium is linear under loading. Then (4.8) is reduced to Eqs.

$$f_1(\varepsilon_{10}) - f(\varepsilon_0) = (V^\circ - v^\circ)h_1', \quad (\varepsilon_{10} - \varepsilon_0)\, h_1' = V^\circ - v^\circ, \quad f_1(\varepsilon_{10}) = V^\circ (\rho_2 a_2 / \rho_1 a_1) \quad (5.1)$$

Here a_2 denotes the constant velocity of propagation of disturbances in the second medium.

Let us introduce the notation

$$X = \varepsilon_{10} - \varepsilon_0, \qquad Y = f_1(\varepsilon_{10}) - f(\varepsilon_0), \qquad h_1' = -\sqrt{\operatorname{tg} \alpha}$$

Eliminating $V^\circ - v^\circ$ and $f_1(\varepsilon_{10})$ from (5.1) we obtain

$$X = \sigma^\circ \frac{\varkappa - 1}{(\varkappa_0 + \sqrt{\operatorname{tg}\alpha})\sqrt{\operatorname{tg}\alpha}}, \qquad Y = \sigma^\circ \frac{(\varkappa - 1)\sqrt{\operatorname{tg}\alpha}}{\varkappa_0 + \sqrt{\operatorname{tg}\alpha}}$$

$$\varkappa_0 = \frac{\rho_2 a_2}{\rho_1 a_1}, \qquad \varkappa = \frac{\rho_2 a_2}{\rho_1 a}, \qquad \sigma^\circ = \frac{\sigma_0(0)}{\rho_1 a_1^2} \qquad \left(a = \frac{a_1 \sigma^\circ}{v^\circ}\right) \qquad (5.2)$$

Here a is the velocity of the incident shock wave at the instant of its collision with the interface.

A descriptive geometrical interpretation of Eqs. (5.2) is presented in Fig. 3. The equations give a parametric representation of line PQ in the system of coordinates XY. It is easily verified that the line PQ represents a monotonously decreasing function the value of which for $\alpha = \pi/2$ is $\sigma^\circ(\varkappa - 1) > 0$. For $\alpha \to 0$ the function approaches zero. Consequently, there is in this case also only one intersection of curve PQ with the diagram of compression MN. Knowing this point we find the desired elements of motion

$$h_1' = -\sqrt{\operatorname{tg}\alpha}, \qquad \varepsilon_{10} = \varepsilon_0 + X,$$

$$V^\circ - v^\circ = -(\varepsilon_{10} - \varepsilon_0)\sqrt{\operatorname{tg}\alpha}$$

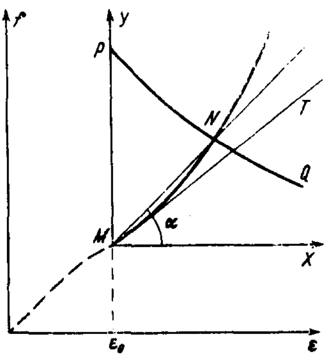

Fig. 3

This not only proves the possibility of unique solution of system (5.1) but also gives a method for numerical determination of desired quantities. This method can be improved through the application of a moving coordinate system XY on transparent paper on which a family of PQ lines can also be drawn.

An important characteristic will be the coefficient of reflection

$$K = \frac{\sigma_1(h_0, t_0)}{\sigma(h_0, t_0)}$$

In the formulation of the problem it was assumed that $K > 1$. Conditions under which this is materialized can now be pointed out. From (5.2) we obtain

$$K = \frac{\varkappa_0 + \varkappa\sqrt{\operatorname{tg}\alpha}}{\varkappa_0 + \sqrt{\operatorname{tg}\alpha}} \tag{5.3}$$

The inequality $K > 1$ indicates, as we see, that

$$a_2\rho_2 > a\rho_1 \tag{5.4}$$

in agreement with [2] where an approximate method was suggested for determination a.

The proof for a unique solution of system (4.8) can be extended at the expense of some complications to the case when the second medium is plastic with an arbitrary compression relationship which is subject only to limitations of a general character. Let us assume that $\epsilon^{-1}f_2(\epsilon)/\epsilon$ is a nondecreasing function. Then we can prove h_2' as a function of V^0 determined by the following Eq.

$$h_2' = \frac{V^\circ}{\varepsilon_{20}} = \frac{a_2}{a_1}\sqrt{\frac{f_2(\varepsilon_{20})}{\varepsilon_{20}}}$$

is also a nondecreasing function. The following estimates are valid

$$h_2' = \frac{a_2}{a_1}\varphi(V^\circ) \geqslant \frac{a_2^\circ}{a_1}, \qquad a_2^\circ = a_2\sqrt{f_2'(0)}$$

Here a_2^0 is the velocity of propagation of small perturbations in the second medium.

The first three Eqs. of system (4.8) can be written in the form (5.5)

$$Y = (v^\circ - V^\circ)\sqrt{\operatorname{tg}\alpha}, \quad X = (v^\circ - V^\circ)/\sqrt{\operatorname{tg}\alpha}, \quad (v^\circ - V^\circ)\sqrt{\operatorname{tg}\alpha} + \sigma^\circ = \varkappa_0 V^\circ \varphi(V^\circ)$$

We shall assume again that the second medium is "more rigid" than the first in the sense that

$$\rho_2 a_2^\circ / \rho_1 a > 1 \tag{5.6}$$

Then for a fixed value of α $(0 \leqslant \alpha \leqslant 1/2\,\pi)$ the third Eq. of system (5.5) determines the only value of V^0 on the section $0 \leqslant V^\circ \leqslant v^\circ$. The reflection coefficient in this case turns out to be greater than one. This is most easily seen from Fig. 4 on which OC is the graphic representation of the right-hand side of the equation while the straight lines mn, $m'n'$ and $m''n''$ represent the left-hand side for various fixed values of α. The point P is located below the curve OC, this follows from the inequality

$$\frac{\varkappa_0 V^\circ \varphi(V^\circ)}{\sigma^\circ} \geqslant \frac{\rho_2 a_2^\circ}{\rho_1 a} > 1$$

This insures intersection in the region $0 < V^0 < v^0$. Eqs.

$$X = \frac{\varkappa\varphi(V^\circ) - 1}{(\varkappa_0 + \sqrt{\operatorname{tg}\alpha})\sqrt{\operatorname{tg}\alpha}}\sigma^\circ, \qquad Y = \frac{(\varkappa\varphi(V^\circ) - 1)\sqrt{\operatorname{tg}\alpha}}{\varkappa_0 + \sqrt{\operatorname{tg}\alpha}}\sigma^\circ \tag{5.7}$$

as previously, give a parametric representation of the line in the XY plane. This line shows a monotonously decreasing function since for a variation of α from $\frac{1}{2}\pi$ to 0 the X coordinate increases (this follows from the second Eq. of (5.5)), while Y decreases. This is evident from the second Eq. of (5.7). Therefore the line (5.7) has a single point of intersection with the curve

$$Y = f_1(X + \varepsilon_0) - f_1(\varepsilon_0)$$

which was to be proved.

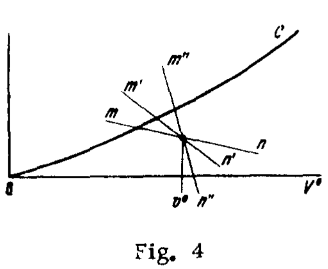

Fig. 4

For the reflection coefficient we obtain

$$K = \frac{\varkappa_0 + \varkappa\varphi(V^\circ)\sqrt{\operatorname{tg}\alpha}}{\varkappa_0 + \sqrt{\operatorname{tg}\alpha}} > 1 \qquad \left(\varkappa\varphi(V^\circ) \geqslant \frac{\rho_2 a_2^\circ}{\rho_1 a} > 1\right)$$

6. For purposes of computing the quantitative aspect of effects for which the theory was presented in Sections 3 to 5, the numerical solution of the problem of reflection of a plastic wave was carried out. The case of reflection from a rigid wall was examined here.

The working region of the law of compression was described by the function $\sigma = \rho_1 a_1^2 \varepsilon^m$. External stress in the function of time was given by

$$\sigma_0(t) = \sigma_0(0)\begin{cases} 0 & \text{for } t < 0,\ t > \theta \\ (1 - t/\theta)^n & \text{for } 0 < t < \theta \end{cases}$$

The following quantities were nondimensional parameters of the problem

$$m,\ n,\ \sigma^\circ = \frac{\sigma_0(0)}{\rho_1 a_1^2},\quad s = \frac{a_1\theta}{h_0},\quad h^\circ = \frac{h}{h_0},\ t^\circ = \frac{t}{\theta}$$

For numerical computation the region of propagation $0 \leq h^0 \leq 1$ is devided into N equal parts ($N = 50$ was selected). Eqs. (2.1) of the incident wave is integrated numerically by Euler's method. Since the desired functions change quite smoothly, this method gave sufficient accuracy. The reflected wave was computed by two methods. The first method was based on the hypothesis of unloading and was reduced to numerical integration of the two Eqs. of system (4.3).

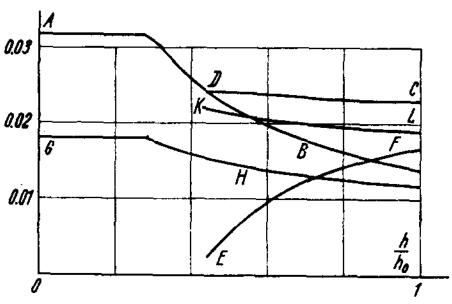

Fig. 5

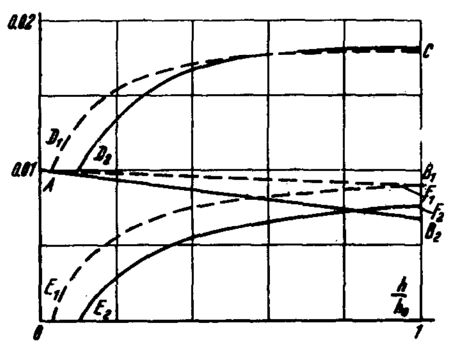

Fig. 6

At the same time a numerical solution was carried out for the correctly formulated problem in which the unloading regime was not assumed in advance. The selection of the regime was accomplished in the process of calculation. Eqs. (1.1) were replaced by difference equations. Boundary conditions were given on the reflecting wall $v = 0$ and on the shock wave.

Below the results of calculations for the following three combinations of parameters are presented

	σ	θ	n	m
1	10^{-3}	1.7	0	2
2	10^{-2}	50	10	2
3	10^{-2}	50	10	1

Results of calculations for the first variant are presented in Fig. 5 where the following functions are given in their dependence on h: ϵ is curve AB; $h_1' 10^{-1}$ is curve KL; ϵ_1 is curve CD; $v_3 10$ is curve EF and $h' 10^{-1}$ is curve GH.

The rise in the propagation velocity with time and the increase of deformation stress on the shock wave may be noted. It was shown by the computation that between the shock wave and the wall an increase of stress takes place up to the moment of disappearance of the shock wave. The increase of these values is quantitatively not large, namely, from the moment of reflection to the exhaustion of the shock wave its velocity of propagation increases approximately by 12% , the deformation by 5% and consequently the stress by 8% . The formal calculation based on the hypothesis of unloading yields quite small differences, namely, it exagerates the value of h_{1*}' by 1% , and the value of ϵ_1 by 4% near the instant of wave extinction. The distribution of values of σ and v between the reflecting wall and the shock wave differs very little from the result which followed from the hypothesis of unloading. In the latter case $\sigma(h, t) = \sigma(t)$ and $v(h, t) \equiv 0$ must hold. In the correct calculation, the deviation of $\sigma(h, t)$ from the constant value along the coordinate does not exceed 2% . The value $v(h, t)$ does not exceed $0.13\, v(t_0)$.

Results of calculations for variants 2 and 3 are shown in Fig. 6 (the graphs for variant 3 are shown dashed) in the form of curves giving the dependence on h for the following quantities $\varepsilon\ (AB_1)$, $\varepsilon^2\ (AB_2)$, $\varepsilon_1\ (CD_1)$, $\varepsilon_1^2\ (CD_2)$, $v_3\ (E_1F_1)$, $v_3 / \sqrt{10}\ (E_2F_2)$

Both these variants are characterized by external interaction which is more prolonged than in variant 1 . This has the result that the reflected shock wave is exhausted later. Velocities h_*' and h_{1*}' are not shown in Fig. 6 because they change little during propagation. For variant 2 the increase in velocity of the reflected wave is here quite small; it represents no more than 1% and in the following is replaced by a deceleration. The values h_{1*}' and ϵ_1 computed from the correct theory and from the hypothesis of unloading practically do not differ (they agree with accuracy to four significant figures). In variant 2 immediately after reflection the loading regime begins which however is replaced in the following by unloading. This is presented in Fig. 7 in which this phenomenon is shown in the plane of variables h/h_0 and t/θ . The line PO_1 is the hodograph of the incident wave, O_1R of the reflected wave; the region RO_1S is the region of loading, QRS is the region of unloading. The relative difference between the maximum and the minimum value of $\sigma(h, t)$ in this region represents less than 0.5% .

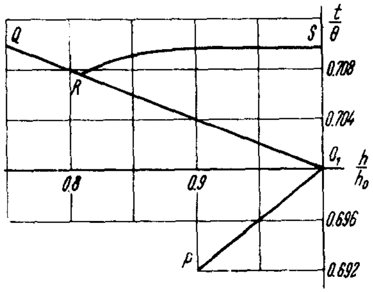

Fig. 7

As a result we can form conclusions with regard to the phenomenon of reflection-refraction of plastic waves within the framework of the formulation of the problem accepted so far in case when the external loading has the nature of an explosive interaction, the unloading is assumed to be rigid and the coefficient of reflection is greater than unity.

1) If the incident wave is nonstationary, the reflected shock wave does not reach the boundary plane on which the external pressure is applied. Extinction of the shock wave occurs the later, the closer the incident wave is to the stationary wave.

2) Under quite general assumptions about compressibility relationships in both media, the system (4.8) of finite nonlinear equations, to which the problem of reflection-refraction of a stationary wave is reduced, has a solution which is in this case unique for the assumed regime of $K > 1$. Recommendations are made for graphic-analytical construction of solution.

3) At the start of reflection the velocity field of the precursor coincides with the

velocity field of the incident wave with accuracy to small quantities of second order (see Section 3, Eq. (3.7)). This is the basis of solution of the problem in [1 to 3] where the effect of the boundary plane $h = 0$ was not taken into account (*). Such solutions naturally have limited significance as approximations for the initial stage after reflection. They describe the phenomenon better for long waves than for short ones. However, these solutions give exactly the quantities related to the instant of collision of the incident wave with the obstacle (see also [4]).

4) The generally accepted a priori hypothesis about the regime of unloading in the region of the reflected wave is generally incorrect. Its incorrectness was proved for the case when the compressibility diagram of the first medium is upward concave in the working region, while there is no special restriction placed on the compressibility relationship of the second medium.

5) If the stress diagrams in the first and second medium are linear, a solution exists which is in agreement with the unloading hypothesis.

6) In the determination of desired functions errors which are introduced because of incorrectness of the problem with the unloading hypothesis, turn out to be quantitatively quite small. This is demonstrated by numerical calculation of a series of examples with reflection from a rigid wall. These calculations were carried out for different combinations of input parameters. Two such examples are presented in this paper in Section 6.

7) We note that reflection from a nonstationary barrier at the boundary of two media was not examined here. In this case the character of reflection is different and requires a different approach for the analysis of the phenomenon in its initial stage. However, the two limiting variants when the mass of the barrier becomes zero or infinite are described by the present theory. This forces us to expect that in the reflection from a nonstationary barrier cases may be encountered which are contradictory to the hypothesis of unloading.

The question of how one should reasonably formulate problems on reflection of plastic waves cannot be examined separately from the purpose and the type of materials for which the problem is formulated. To-date such problems have been studied in connection with the needs of construction technology. As a rule in these cases experimental accuracy and the expected accuracy of answers are much rougher than those fine deviations which were discovered by our calculations. Many initial assumptions about rigidity of unloading, about approximation of the compressibility diagram and others are made by authors with great ease. This lack of concern is justified by the absence of reliable and stable data on properties of many real materials. Against this background the actual errors introduced by the incorrectness of the unloading hypothesis appear of small significance. Therefore, calculations of reflection and refraction based on the hypothesis of unloading, and consequently incorrect in principle, should be accepted as satisfactory for the solution of many principal problems.

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*) In [1] it was assumed that in all instances of time the velocity field of the precursor coincides with the velocity field of the extended incident wave. This is an independent assumption in spite of an incorrect reference to continuity of displacements (see e.g. [1]).

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Translated by B. D.